DOMINATION NUMBER EXTENSIONS: FROM BASE GRAPHS TO LINE GRAPHS

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ABSTRACT

Let G be a simple connected graph of order n and L(G) be its line graph. A subset S of V is called a dominating set of G if every vertex of V – S is adjacent to some vertex in S. The domination number γ(G) of G is the minimum cardinality taken over all dominating sets of G. In this paper, we characterize regular graphs and unicyclic graphs of odd order for which γ(G) + γ(L(G)) = n − 2.

Keywords: *Domination Number, Line Graph, Unicyclic Graphs, Regular Graphs*.

I. INTRODUCTION

Let G = (V, E) be a connected graph of order n and size m. The undefined terms and notations can be found in [5]. In 1956, Nordhaus and Gaddum [12] gave the lower and upper bound for the sum and product of chromatic number of a graph and its complement. In 1972, Jaeger and Payan [6] proved the same for domination number. The line graph L(G) of a graph whose vertex set is E(G) and two vertices of L(G) are adjacent if and only if the corresponding edges are adjacent in G. The concept of edge domination was introduced by Mitchell and Hedetniemi [10]. A subset S' of E is called an edge dominating set of G if every edge not in S' is adjacent to some edge in S'. The edge domination number γ'(G) of G is the minimum cardinality taken over all edge dominating sets of G. The domination number of a line graph L(G) of a graph G is the same as an edge domination number of a graph, that is γ'(G) = γ(L(G)). Recently [11], the authors characterized lower and upper bound for the sum γ(G) + γ(L(G)). In this paper, we characterize γ(G) + γ(L(G)) = n − 2 for regular graphs and unicyclic graphs of odd order.

II. PRELIMINARY RESULTS

The following results are required for our main theorems.

Theorem 2.1. ([4,13]) For a graph G with even order n and no isolated vertices, γ(G) = n / 2 if and only if the components of G are the cycle C_4 or the corona H o K_1 for any connected graph H.

In [3] E. J. Cockayne, et al characterized connected graphs for which γ(G) = $\lfloor n/2 \rfloor$. For this characterization, they defined six classes of graphs by using the following families of graphs. Let

$$\mathcal{G}_1 = \{C_4\} \cup \{G : G = H \circ K_1, \text{ where H is connected}\}$$

and

$$\mathcal{G}_2 = \mathcal{A} \cup \mathcal{B} - \{C_4\}$$

For any graph H, $\mathcal{S}$(H) denote the set of connected graphs, each of which can be formed from H o K_1 by adding a new vertex x and edges joining x to one or more vertices of H. Then define

$$\mathcal{G}_3 = \cup_H \mathcal{S}(H)$$

Figure : 1

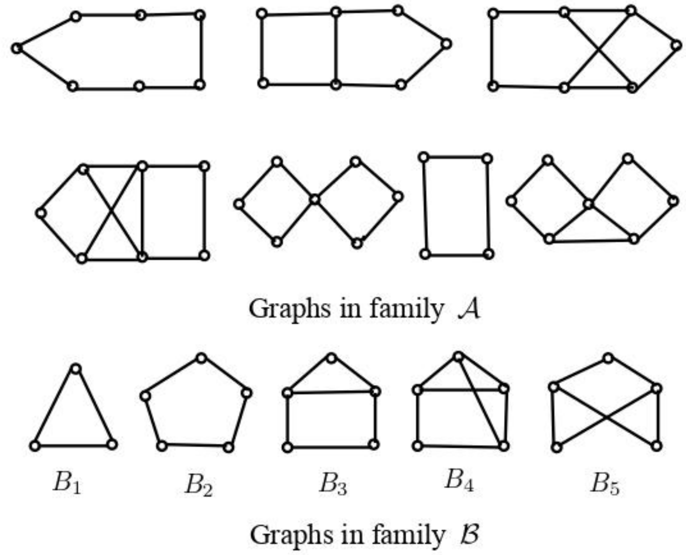

Fig. 1.

where the union is taken over all graphs H. Let y be a vertex of a copy of C_4 and, for G $\in \mathcal{G}_3$, let θ(G) be the graph obtained by joining G to C_4 with the single edge xy, where x is the new vertex added in forming G. Then define

$$\mathcal{G}_4 = \{\theta(G) : G \in \mathcal{G}_3\}$$

Next, let u, v, w be a vertex sequence of a path P_3. For any graph H, let $\mathcal{P}$(H) be the set of connected graphs which may be formed from H o K_1 by joining each of u and w to one or more vertices of H. Then define

$$\mathcal{G}_5 = \cup_H \mathcal{P}(H)$$

Let H be a graph and X $\in \mathcal{B}$. Let $\mathcal{R}$(H, X) be the set of connected graphs which may be formed from H o K_1 by joining each vertex of $U \subseteq V(X)$ to one or more vertices of H such that no set with fewer than γ(X) vertices of X dominates V(X) – U. Then define

$$\mathcal{G}_6 = \cup_{H,X} \mathcal{R}(H, X).$$

Theorem 2.2.([3]) A connected graph G satisfies $\gamma(G) = \lfloor n/2 \rfloor$ if and only if $G \in \mathcal{G} = \cup_{i=1}^{6} \mathcal{G}_i$.

Theorem 2.3.([14]) If G is a connected graph with $\delta(G) \geq 3$, then $\gamma(G) \leq (3n)/8$.

Theorem 2.4.([2]) For any graph G, $\lceil n/(1 + \Delta(G)) \rceil \leq \gamma(G) \leq n - \Delta(G)$.

Theorem 2.5.([8]) If a graph G has no isolated vertices and $\gamma(G) \geq 3$, then $\gamma(G) \leq (n + 1 - \delta)/2$.

Theorem 2.6.([1]) For any connected graph G of even order n, $\gamma'(G) = n/2$ if and only if G is isomorphic to K_n or $K_{n/2, n/2}$.

The graph obtained by identifying the centre of a subdivided star $S(S_{1, k})$ with a vertex of C_3 is denoted by $C_{3, k}$.

The graph obtained by joining the centre of subdivided star $S(S_{1, k})$ with a vertex of C_4 by an edge e is denoted by $C_{4, k}(e)$.

Theorem 2.7.([1]) Let G be a connected unicyclic graph. Then $\gamma'(G) = \lfloor n/2 \rfloor$ if and only if G isomorphic to C_4, C_5, C_7, $C_{3,k}$, $C_{4,k}$ (e) for some $k \geq 0$.

Lemma 2.8 Let H be any subgraph of G. Then $\gamma(G) \leq \gamma(H) + \gamma(G - V(H))$.

Lemma 2.9 If H is a subgraph of G, then $\gamma'(G) \leq \gamma'(H) + \gamma'(G - E(H))$.

Notation 2.10 ([7]) If G is a graph with vertex set V = $\{u_1, u_2,\ldots\}$, then the graph obtained by identifying one of the end vertices of n_2 copies of P_2, n_3 copies of P_3 at u_1, m_2 copies of P_2, m_3 copies of P_3…. at u_2…. is denoted by $G[u_1(n_2P_2, n_3P_3,\ldots); u_2(m_2P_2, m_3P_3,\ldots); \ldots]$.

III. MAIN RESULTS

Theorem 3.1 Let G be a connected unicyclic graph of odd order $n \geq 5$. Then $\gamma(G) + \gamma(L(G)) = n - 2$ if and only if G is isomorphic to one of the graphs $G_1, G_2, \ldots, G_{29}$ given in Figure 2.

Proof: Let G be a connected unicyclic graph of odd order $n \geq 5$. If $\gamma(G) + \gamma(L(G)) = n - 2$, then we have the following two cases.

Case: 1 $\gamma(G) = (n - 3)/2$ and $\gamma'(G) = (n - 1)/2$.
By Theorem 2.7, G isomorphic to C_5, C_7, $C_{3,k}$, $C_{4,k}(e)$ for some $k \geq 0$. But $\gamma(G) = (n - 1)/2$ for these graphs.

Case: 2 $\gamma(G) = (n - 1)/2$ and $\gamma'(G) = (n - 3)/2$.
By Theorem 2.2, $G \in \bigcup_{i=2}^{6} \mathcal{G}_i$. If $G \in \mathcal{G}_2$, then it is easy to verify that $\gamma(L(G)) = \gamma'(G) = (n - 1)/2$ for these graphs.

Subcase: 2.1 $G \in \mathcal{G}_3$
If H is connected, then by Lemma 2.9, diam(H) = 1 or 2 and so H is either K_2 or star or C_3 or C_4 or C_5 or $C_3[u(kP_2)]$. If H is a star, then x is adjacent to exactly two vertices of H. Hence G is isomorphic to G_1 or G_2 which satisfy the hypothesis. If H = C_3 or C_4 or C_5 or $C_3[u(kP_2)]$, then x is adjacent to exactly one vertex of H. When H = C_4 or C_5 or $C_3[u(kP_2)]$, we observe that, $\gamma'(G) = (n - 5)/2 \neq (n - 3)/2$. If H = C_3, then G is isomorphic to G_3 which satisfy the hypothesis. If H is disconnected, let $H_1, H_2,\ldots, H_s$ be the components of H. Clearly exactly one component, say H_i is nontrivial and diam(H_i) = 1 or 2 and other H_j's ($j \neq i$) are trivial. Then by the previous argument for H_i, G is isomorphic to G_4 or G_5 or G_6 which satisfy the hypothesis.

Subcase 2.2 $G \in \mathcal{G}_4$
If H is connected, then we observe that diam(H) = 1 or 2. By the definition of $\mathcal{G}_4$, H must be either K_2 or star. Hence G is isomorphic to G_7 or G_8 which satisfy the hypothesis. If H is disconnected, then exactly one of its components is non-trivial whose diameter is 1 or 2 and others are trivial. Hence G is isomorphic to G_9 or G_{10} which satisfy the hypothesis.

Subcase 2.3 $G \in \mathcal{G}_5$
If H is connected, then by Lemma 2.9, diam(H) = 1or 2 and so H is a star or C_3 or C_4 or C_5 or $C_3[u(kP_2)]$. If H is a star, then both u and w are adjacent to exactly one vertex of H (or) u and w are adjacent to two distinct vertices of H. Hence G is isomorphic to G_{11}, G_{12} or G_{13} which satisfy the hypothesis. If H = C_3 or C_4 or C_5 or $C_3[u(kP_2)]$, then either u or w is a pendant vertex (say w). When H = C_4, C_5 or $C_3[u(kP_2)]$, we observe that, $\gamma'(G) = (n - 5)/2 \neq (n - 3)/2$. If H = C_3, then $G \in G_6$.
If H is disconnected, let $H_1, H_2,\ldots, H_s$ be the components of H. Then we have the following two cases.

Case: 2.3.1 Either u or w is a pendant vertex (say w).
Then exactly one component, say H_i is non-trivial and diam(H_i) = 1 or 2 and other H_j's ($j \neq i$) are trivial. If H_i is a star, then u is adjacent to exactly two vertices of H_i and it is adjacent to each vertex of $H_j = K_1$ ($j \neq i$). Hence G is isomorphic to G_4 or G_5. Clearly $H_i = C_3$; otherwise $\gamma'(G) < (n - 3)/2$. If $H_i = C_3$, then u is adjacent to exactly one vertex to each component of H. Hence G is isomorphic to G_6.

Case: 2.3.2 Both u and w are not pendant vertices.
Then $< uvw >$ is isomorphic to C_3 or P_3. If $< uvw >$ is isomorphic to C_3, then u and w are adjacent to different components of H and H_i must be a tree. If H_i is trivial, then G is isomorphic to G_{14} for which $\gamma'(G) = (n - 3)/2$. If diam(H_i) $\leq$ 2, then H_i is a star and G is isomorphic to G_{15} or G_{16} which satisfy the hypothesis. Now let $< uvw >$ be isomorphic to P_3. If H = C_3 or C_4 or C_5 or $C_3[u(kP_2)]$, then by Lemma 2.9, $\gamma'(G) < (n - 3)/2$. Hence H_i must be a tree and note that diam(H_i) = 0 or 1 or 2. Since H_i is a tree, both u and w are adjacent to exactly one vertex of H_i (or) u and w are adjacent to two distinct vertices of H_i. If diam(H_i) = 0, ($H_i = K_1$ for all i), then G is isomorphic to G_{17}. If diam (H_i) $\leq$ 2, then H_i is a star. Hence G must be one of the graphs G_{18}, G_{19}, G_{20} which satisfy $\gamma'(G) = (n - 3)/2$.

Subcase 2.4 $G \epsilon \mathcal{G}_6$

By the definition of $\mathcal{G}_6$, X must be C_3 or C_5.

Case 2.4.1: $X = C_3$.

If H is connected, then H is a tree with diam(H) ≤ 2 and so H is either K_1 or a star $K_{1,r}$($r \geq 1$). Clearly |V(U)| must be 1. If $H = K_1$, then G is isomorphic to $C_{3,1}$ but $\gamma'(G) = (n-1)/2 \neq (n-3)/2$. If H is a star, then G is isomorphic to G_{21} or G_{22} which satisfy the hypothesis. If H is disconnected, then it is either totally disconnected or exactly one component, say H_i is of diameter at most 2 and other components H_j's ($j \neq i$) are trivial. It is clear that $H = K_1$ or a star and |V(U)| = 1 or 2. Suppose |V(U)| = 1. If $H_i = K_1$, then G is isomorphic to $C_{3,k}$ but $\gamma'(G) = (n-1)/2 \neq (n-3)/2$. If H_i is a star, then G is isomorphic to G_{23} or G_{24}. Suppose |V(U)| = 2. If $H_i = K_1$, then G is isomorphic to G_{14}. If H_i is a star, then G is isomorphic to G_{15} or G_{16}.

Figure : 2

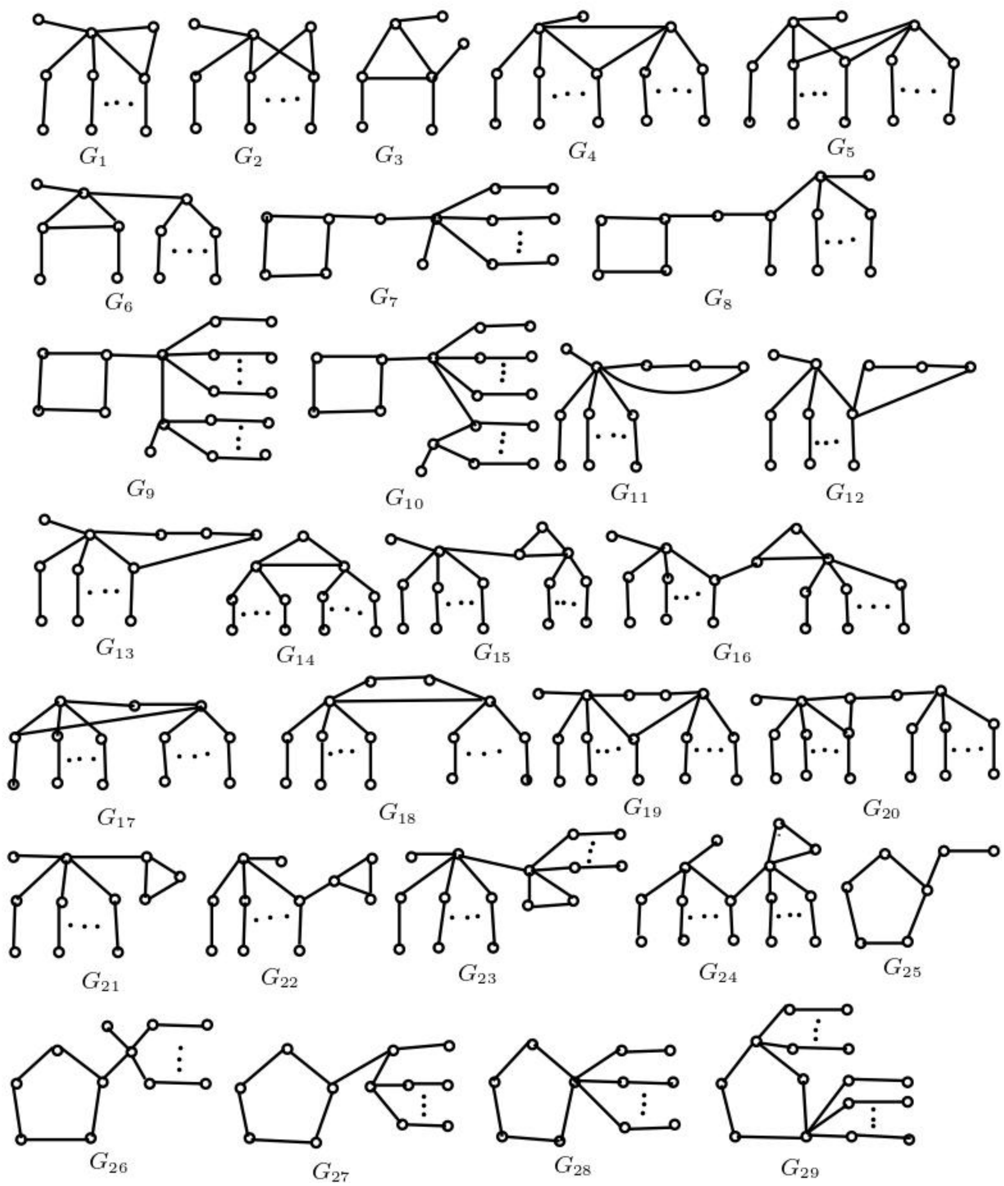

Fig. 2 Unicyclic Graphs satisfying γ(G) + γ(L(G)) = n − 2

Case 2. 4. 2: $X = C_5$.

If H is connected, then H is a tree with diam(H) ≤ 2 and |V(U)| = 1. If $H = K_1$, then G is isomorphic to G_{25}. If H is a star, then G is isomorphic to G_{26} or G_{27}. If H is disconnected, then each component of H is trivial and |V(U)| = 1 or 2. If |V(U)| = 1, then G is isomorphic to G_{28}. |V(U)| = 2, then G is isomorphic to G_{29}.

Conversely, if G is isomorphic to G_1, G_2, …, G_{29}, then it can be easily verified that γ(G) + γ(L(G)) = n − 2.
The following lemma 3.2 is useful for Theorem 3.3.

Lemma 3.2 If G is a connected 4-regular graph of order 11, then γ(G) = 3.

Proof : Let G be a connected 4-regular graph of order 11. Clearly γ(G) ≥ 3. Let S be a γ-set of G. Let v be an arbitrary vertex of G and $N(v) = \{v_1, v_2, v_3, v_4\}$. Consider G' = G − N[v]. Since |V(G')| = 6, let $V(G') = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$. Clearly γ(G') ≠ 1. Let S' be a γ-set of G'. If |S'| = 2, then S' ∪ {v} is a γ-set of G and hence γ(G) = 3. Let

E_1 denote the set of edges between the vertices of G' and N(v) in G. Since G is 4-regular, $|E_1| \leq 12$ and G' has at most two isolated vertices.

Case 1: G' has two isolated vertices.
Let v_5, v_6 be the two isolated vertices in G' which are adjacent to all the vertices of N(v) in G. Then the remaining components of G' are either $2K_2$ or $K_3+\{e\}$ or C_4 or $C_4+\{e\}$ or $K_{1,3}$ or P_4 or K_4. Since $|E_1| = 12$, G' = $K_4 \cup 2K_1$ and v_1 is adjacent to both v_5 and v_6. Then S = $\{v, v_1, v_7\}$ is a minimum dominating set of G and hence $\gamma(G) \leq 3$ and so $\gamma(G) = 3$.

Case 2 : G' has one isolated vertex (say, v_5).
Then G' is isomorphic to $H \cup K_1$, where $|V(H)| = 5$. Since $|E_1| \leq 12$, H is connected and has at most one pendant vertex and $\Delta(H) = 3$ or 4. If $\Delta(H) = 4$, let $d(v_6) = 4$ in G'. Then S = $\{v, v_5, v_6\}$ is a minimum dominating set of G and hence $\gamma(G) = 3$. Now consider the case for $\Delta(H) = 3$. Let $d(v_6) = 3$ in H and $V(H - N[v_6]) = v_{10}$. Clearly v_{10} must be adjacent to at least one of the vertices of N(v), (say v_1) in G. Then S = $\{v, v_1, v_6\}$ is a minimum dominating set of G and hence $\gamma(G) \leq 3$ and so $\gamma(G) = 3$.

Case 3 : G' has no isolated vertices.
Since $|E_1| \leq 12$, G' is connected and is isomorphic to $C_3 \circ K_1$. Let $V(C_3) = \{v_5, v_6, v_7\}$ and v_8, v_9, v_{10} be the corresponding pendant vertices of v_5, v_6, v_7 in G'. Since at least one N(v), say v_1 must be adjacent to v_8, v_9, v_{10}. Then S = $\{v, v_1, v_5\}$ is a minimum dominating set of G and hence $\gamma(G) \leq 3$ and so $\gamma(G) = 3$. This completes the proof.

Theorem 3.3 Let G be a connected k-regular graph of order $n \geq 5$. Then $\gamma(G) + \gamma(L(G)) = n - 2$ if and only if G is isomorphic to either K_5, K_6 , $K_{4,4}$ or any one of the graphs F_1, F_2 given in Figure 3.

Figure : 3

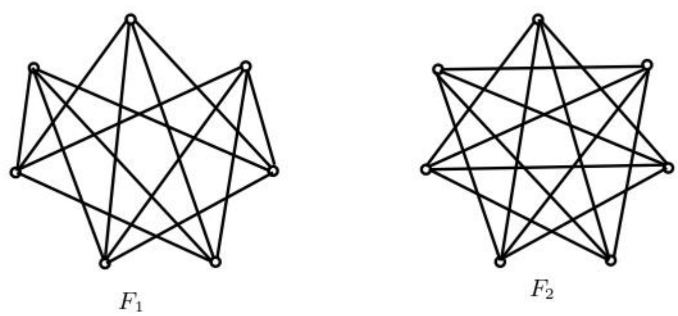

Fig. 3 Regular graphs satisfying $\gamma(G) + \gamma(L(G)) = n - 2$

Figure : 4

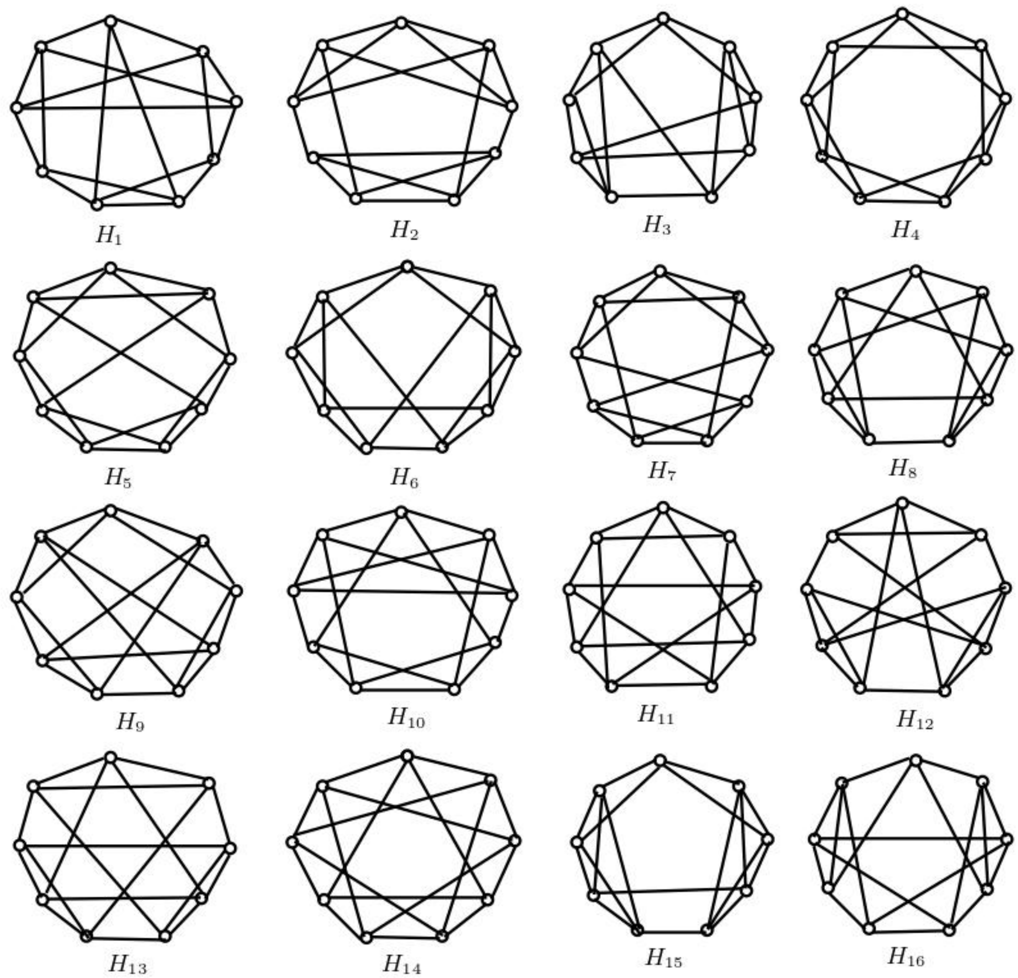

Fig. 4 All 4-regular connected graphs of order 9

Proof: Assume that $\gamma(G) + \gamma(L(G)) = n - 2$. Then we have the following two cases.

Case 1: n is even

If $\gamma(G) = n/2$ and $\gamma'(G) = (n/2) - 2$, then by Theorem 2.1 and hypothesis, no graph exists. If $\gamma(G) = (n/2) - 2$ and $\gamma'(G) = n/2$, then by Theorem 2.6, G is isomorphic to K_n or $K_{n/2, n/2}$. If $G = K_n$, then $\gamma(G) = 1 = (n/2) - 2$ which gives n = 6 and hence $G = K_6$. If $G = K_{n/2, n/2}$, then $\gamma(G) = 2 = (n/2) - 2$ which gives n = 8 and hence $G = K_{4,4}$.

Case 2: n is odd

If $\gamma(G) = (n-1)/2$ and $\gamma'(G) = (n-3)/2$, then by Theorem 2.2, G is either C_5 or C_7 for which $\gamma'(G) = (n-1)/2 \neq (n-3)/2$. Now we consider the case $\gamma(G) = (n-3)/2$ and $\gamma'(G) = (n-1)/2$. If G is 2-regular, then $G = C_n$. We observe that $\gamma'(C_n) = \lceil n/3 \rceil = (n-1)/2$ which gives n = 5 and hence $G = C_5$ but $\gamma(G) = 2 \neq (n-3)/2$. If $k \geq 3$, then by Theorem 2. 3, $n \leq 12$. Since n is odd, k must be even and by hypothesis, $n \in \{5, 7, 9, 11\}$. If n = 5, then $G = K_5$ for which $\gamma(G) = 1 = (n-3)/2$ and $\gamma'(G) = 2 = (n-1)/2$. If n = 7, then k must be either 4 or 6. If k = 6, then $G = K_6$ for which $\gamma(G) = 1 \neq (n-3)/2$. If k = 4, then by [8], there are exactly two graphs F_1, F_2 which are satisfy the hypothesis. If n = 9, then k = 4 or 6 or 8. If k = 4, then by [8], there are sixteen 4-regular graphs of order 9 (See Figure 4) and it is easy to see that no graph satisfies the hypothesis. If k = 6, then by Lemma 2.8, $\gamma(G) < (n-3)/2$, a contradiction. If k = 8, then $G = K_9$ for which $\gamma(G) = 1 \neq (n-3)/2$. If n = 11, then k = 4 or 6 or 8 or 10. If k = 6 or 8 or 10, then it is easy to see that $\gamma(G) < (n-3)/2$. If k = 4, then by Lemma 3.2, $\gamma(G) = 3 \neq (n-3)/2$. Converse is obvious by verification.

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