THE ENDOWMENT STRUCTURE OF TYPE-III UNICYCLIC GRAPHS: A COMPREHENSIVE STUDY

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ABSTRACT

Let G = (V, E) be a non empty, finite, simple graph. A dominating set of a graph G containing a minimum dominating set of G is called a γ - endowed dominating set of G. If that set is of cardinality *k* then it is called a *k*γ – endowed dominating set. *k* - $\gamma_r$enresdowed graph is one in which every restrained dominating set of cardinality *k* contains a minimum restrained dominating set. Consider a cycle graph G, in which a set of different paths is attached to every vertex of the cycle. In this paper, the enresdowedness property for the unicyclic graphs with a set of different paths attached to every vertex of the cycle is obtained.

Keywords: *Enresdowed graphs, Unicyclic graphs.*

I. INTRODUCTION

Let G = (V, E) be a non empty, finite, simple graph. A subset D of V(G) is called a dominating set of G if for every v ∈ V – D, there exists u ∈ D such that u and v are adjacent. The minimum cardinality of the dominating set is called the domination number and it is denoted by γ(G). The restrained dominating set of a graph is a dominating set in which every vertex in V – D is adjacent to some other vertex in V – D[6]. The minimum cardinality of the restrained dominating set is called the restrained domination number and it is denoted by γ_r(G). A graph is said to be *k* - γ_r enresdowed graph if every restrained dominating set of cardinality *k* contains a minimum restrained dominating set. Consider a unicyclic graph G which contains a cycle C_n , n≥ 3, and a set of paths P_t, t≥ 2, where these set s, s≥ 1of different paths are attached to each vertex of C_n.Anders Sune Pedersen,PrebenDahlVestergaardobtained the upper and lower bounds for the number of independent sets in a unicyclic graph in terms of its order[1].A unicyclic graph is a connected graph with a unique cycle. A unicyclic graph is called fully loaded if every vertex on its unique cycle has degree at least three[7]. Joanna Raczek characterize all connected unicyclic graphs with the domination multisubdivision number equal to three[2].

II. RESULTS ON TYPE – III UNICYCLIC ENRESDOWED GRAPHS

Definition 2.1

Let *k* be a positive integer. A simple, finite, non trivial graph G = (V, E) is called a *k* - γ_r enresdowed graph if every restrained dominating set of G of cardinality *k* contains a minimum restrained dominating set γ_r of G.[5]

Definition 2.2

Let G be a unicyclicgraph C_nP_t , for n≥ 3, t≥ 2. Let $\{v_i\}$,1 ≤ i ≤ n be the set of vertices of C_n. The graph G contains a set of n copies of distinct paths $\{P_{it_j}\}$,1 ≤ i ≤ nand 2 ≤ j ≤ s_i, which are attached to each vertex $\{v_i\}$,1 ≤ i ≤ n of the cycle C_n , for n≥ 3. The set of vertices $\{v_i\}$,1 ≤ i ≤ n is considered as the initial vertex for the set of all paths $\{P_{it_j}\}$1 ≤ i ≤ n, 2 ≤ j ≤ s_i attached to each $\{v_i\}$,1 ≤ i ≤ n.

Theorem 2.3

Let G be a unicyclicgraph C_nP_t , for n≥ 3 and t≥ 2. Let $\{v_i\}$,1 ≤ i ≤ n be the set of vertices of C_n. The graph G contains a set of n copies of distinct paths $\{P_{it_j}\}$, 1 ≤ i ≤ n and 2 ≤ j ≤ S_i, which are attached to each vertex $\{v_i\}$, 1 ≤ i ≤ n of the cycle C_n , for n≥ 3, such that the cardinality of any path P_{it_j}, 1 ≤ i ≤ n, 2 ≤ j ≤ S_i is not same as any other path $P_{it_{j+1}}$, for 1 ≤ i ≤ n, 2 ≤ j ≤ S_i which are attached to same v_i, 1 ≤ i ≤ n in G. The set of vertices $\{v_i\}$,1 ≤ i ≤ n of the cycle C_n , for n≥ 3 is considered as the initial vertices for the set of all paths $\{P_{it_j}\}$,1 ≤ i ≤ n,

$2 \leq j \leq S_i$, attached to it. Let D be the minimum restrained dominating set of G, then G is k - γ_r enresdowed for any k , where $\gamma_r \leq k \leq n + \left| \bigcup^{n}_{\substack{i=1 \\ 2 \leq j \leq S_i}} P_{it_j} - S_i \right|$

Proof

Given G is a unicyclicgraph C_nP_t , for $n \geq 3$ and $t \geq 2$. Let $\{v_1, v_2, v_3, \ldots, v_i, \ldots, v_n\}$, $1 \leq i \leq n$, be the set of all vertices of the cycle C_n, $n \geq 3$. Let P_{1t_2} be a path P_2, which consist of the vertex set $\{u_{1,21}, u_{1,22}\}$, such that the vertex $u_{1,21} = v_1$, P_{1t_3} be a path P_3, with the vertex set $\{u_{1,31}, u_{1,32}, u_{1,33}\}$, where the vertex $u_{1,31} = v_1$, P_{1t_4} be a path P_4, with the vertex set $\{u_{1,41}, u_{1,42}, u_{1,43}, u_{1,44}\}$ such that the vertex $u_{1,41} = v_1$. Without loss of generality, consider any path P_{1t_l}, where P_{1t_l} be a path P_l, $2 \leq l \leq S_1$, which contains the vertex set $\{u_{1,l1}, u_{1,l2}, u_{1,l3}, \ldots, u_{1,ll}\}$ where the vertex $u_{1,l1} = v_1$. Proceeding similarly, consider any path $P_{1t_{S_1}}$, where the path $P_{1t_{S_1}}$ is a path P_{S_1} which consist of the vertex set $\{u_{1,S_11}, u_{1,S_12}, u_{1,S_13}, \ldots, u_{1,S_1S_1}\}$, such that the vertex $u_{1,S_11} = v_1$. Thus the set of vertices $\{u_{1,21}, u_{1,22}, u_{1,31}, u_{1,32}, u_{1,33}, u_{1,41}, u_{1,42}, u_{1,43}, u_{1,44}, \ldots, u_{1,l1}, u_{1,l2}, u_{1,l3}, \ldots, u_{1,ll}, \ldots, u_{1,S_11}, u_{1,S_12}, u_{1,S_13}, \ldots, u_{1,S_1S_1}\}$, belongs to the paths $P_{1t_2}, P_{1t_3}, P_{1t_4}, \ldots, P_{1t_l}, \ldots, P_{1t_{S_1}}$ of G, where these sets of all paths are attached to the vertex v_1 of C_n.

Without loss of generality, consider another new set of paths $P_{2t_2}, P_{2t_3}, P_{2t_4}, \ldots, P_{2t_l}, \ldots, P_{2t_{S_2}}$ which are attached to the vertex v_2 of C_n for $n \geq 3$. Let the path P_{2t_2} be a path P_2, with the vertex set $\{u_{2,21}, u_{2,22}\}$, where the vertex $u_{2,21} = v_2$ and P_{2t_3} be a path P_3, which consist of the vertex set $\{u_{2,31}, u_{2,32}, u_{2,33}\}$, such that the vertex $u_{2,31} = v_2$. Let P_{2t_4} be a path P_4 with the vertex set $\{u_{2,41}, u_{2,42}, u_{2,43}, u_{2,44}\}$ where the vertex $u_{2,41} = v_2$. Without loss of generality consider any path P_{2t_l} where P_{2t_l} be a path P_l, $2 \leq l \leq S_2$, with the set of vertices $\{u_{2,l1}, u_{2,l2}, u_{2,l3}, \ldots, u_{2,ll}\}$, such that the vertex $u_{2,l1} = v_2$. Finally consider any path $P_{2t_{S_2}}$, where the path $P_{2t_{S_2}}$ is a path P_{S_2} which consist of the vertex set $\{u_{2,S_21}, u_{2,S_22}, u_{2,S_23}, \ldots, u_{2,S_2S_2}\}$, such that the vertex $u_{2,S_21} = v_2$. Thus the set of vertices $\{u_{2,21}, u_{2,22}, u_{2,31}, u_{2,32}, u_{2,33}, u_{2,41}, u_{2,42}, u_{2,43}, u_{2,44}, \ldots, u_{2,l1}, u_{2,l2}, u_{2,l3}, \ldots, u_{2,ll}, \ldots, u_{2,S_21}, u_{2,S_22}, u_{2,S_23}, \ldots, u_{2,S_2S_2}\}$ which belongs to the set of paths $\{P_{it_j}\}$, for $i = 2$, and $2 \leq j \leq S_2$.

In general, consider a new set of paths, $P_{it_2}, P_{it_3}, P_{it_4}, \ldots, P_{it_l}, \ldots, P_{it_{S_i}}$, $1 \leq i \leq n$. These paths are attached to the vertex v_i, $1 \leq i \leq n$ of C_n for $n \geq 3$. Let P_{it_2} be a path P_2, with the vertex set $\{u_{i,21}, u_{i,22}\}$, such that the vertex $u_{i,21} = v_i$, $1 \leq i \leq n$. The path P_{it_3} be a path P_3 which consist of the set of vertices $\{u_{i,31}, u_{i,32}, u_{i,33}\}$,where the vertex $u_{i,31} = v_i$, $1 \leq i \leq n$. Let P_{it_4} be a path P_4 with the vertex set $\{u_{i,41}, u_{i,42}, u_{i,43}, u_{i,44}\}$, such that the vertex $u_{i,41} = v_i$, $1 \leq i \leq n$. Similarly consider any path P_{it_l}, where P_{it_l} is a path P_l, $2 \leq l \leq S_i$, $1 \leq i \leq n$ which consist of the set of vertices $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$, such that the vertex $u_{i,l1} = v_i$, $1 \leq i \leq n$. Finally consider any path $P_{it_{S_i}}$, $1 \leq i \leq n$,where the path $P_{it_{S_i}}$ is a path P_{S_i} which consist of the vertex set $\{u_{i,S_i1}, u_{i,S_i2}, u_{i,S_i3}, \ldots, u_{i,S_iS_i}\}$, for $1 \leq i \leq n$, such that the vertex $u_{i,S_i1} = v_i$, $1 \leq i \leq n$. Then the set of vertices $\{u_{i,21}, u_{i,22}, u_{i,31}, u_{i,32}, u_{i,33}, u_{i,41}, u_{i,42}, u_{i,43}, u_{i,44}, \ldots, u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}, \ldots, u_{i,S_i1}, u_{i,S_i2}, u_{i,S_i3}, \ldots, u_{i,S_iS_i}\}$ belongs to the set of paths $P_{it_2}, P_{it_3}, P_{it_4}, \ldots, P_{it_l}, \ldots, P_{it_{S_i}}$, $1 \leq i \leq n$.

Proceeding similarly, consider a new set of paths in G. Let P_{nt_2}, be a path P_2, with the vertex set $\{u_{n,21}, u_{n,22}\}$, where the vertex $u_{n,21} = v_n$. The path P_{nt_3}, be a path P_3, which consist of the set of vertices $\{u_{n,31}, u_{n,32}, u_{n,33}\}$such that the vertex $u_{n,31} = v_n$. Let P_{nt_4} be a path P_4, with the vertex set $\{u_{n,41}, u_{n,42}, u_{n,43}, u_{n,44}\}$, where the vertex $u_{n,41} = v_n$. Proceeding similarly, consider the path P_{nt_l}, where P_{nt_l}, $2 \leq l \leq S_n$ is a path P_l which contains the set of vertices $\{u_{n,l1}, u_{n,l2}, u_{n,l3}, \ldots, u_{n,ll}\}$, such that the vertex $u_{n,l1} = v_n$. Finally, consider the path $P_{nt_{S_n}}$, where the path $P_{nt_{S_n}}$ is a path P_{S_n} with the vertex set $\{u_{n,S_n1}, u_{n,S_n2}, u_{n,S_n3}, \ldots, u_{n,S_nS_n}\}$, where the vertex $u_{n,S_n1} = v_n$. Then the set of vertices $\{u_{n,21}, u_{n,22}, u_{n,31}, u_{n,32}, u_{n,33}, u_{n,41}, u_{n,42}, u_{n,43}, u_{n,44}, \ldots, u_{n,l1}, u_{n,l2}, u_{n,l3}, \ldots, u_{n,ll}, \ldots, u_{n,S_n1}, u_{n,S_n2}, u_{n,S_n3}, \ldots, u_{n,S_nS_n}\}$, belongs to the set of paths $P_{nt_2}, P_{nt_3}, P_{nt_4}, \ldots, P_{nt_l}, \ldots, P_{nt_{S_n}}$, where these set of paths are attached to the vertex v_n of C_n. Thus the graph $G = C_nP_t$, for $n \geq 3$ and $t \geq 2$ is obtained.

Let D be the minimum restrained dominating set of G. The set of all paths $\{P_{it_j}\}$, $1 \leq i \leq n$ and $2 \leq j \leq S_i$ will be of any one of the following types.

Case (i) Suppose if $P_{it_l} = P_{3m-1}$, for m ≥ 1, 1 ≤ i ≤ n , where l = 2,5,8,….., then the path P_{it_l} be P_l, where the vertex set of P_l, be $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$ for 1 ≤ i ≤ n, such that the vertex $u_{i,l1} = v_i$. Without loss of generality, choose the set of vertices $u_{i,l2}$, for 1 ≤ i ≤ n, l = 2,5,8,….. for the γ_r set D from the path of the type P_{3m-1}, for m ≥ 1, from each u_i then the set of vertices $u_{i,l3}$, for l≠2 where l = 5,8,….. are dominated. Similarly choose the set of vertices $u_{i,l5}$, for l = 5,8,….. for the γ_r set D, then the set of vertices $u_{i,l4}$, l = 5,8,….. and $u_{i,l6}$, 1 ≤ i ≤ n,l = 8,….. are dominated, such that the set of vertices $u_{i,l3}$ and $u_{i,l4}$ l = 5,8,….. are adjacent in V – D, similarly choose the set of vertices $u_{i,l8}$ for l≠2,5 , where l = 8,11,14,…..then the set of vertices $u_{i,l7}$, for 1 ≤ i ≤ n and l = 8,11,14,….. are dominated. Hence the set of vertices $u_{i,l6}$, for l = 5,8,….. and $u_{i,l7}$, for 1 ≤ i ≤ n and l = 8,11,14,….. are adjacent in V – D. Thus Proceeding similarly, choose the set of all vertices $\{u_{i,ll}\}$, l = 2,5,8,….. for the γ_r set D, from the path of the type $P_{it_l} = P_{3m-1}$, for m ≥ 1, 1 ≤ i ≤ n. Let $C_1 = \{u_{i,l2}, u_{i,l5}, u_{i,l8}, \ldots, u_{i,ll}\}$ for 1 ≤ i ≤ n, l = 2,5,8,….. be the set of vertices chosen from paths of type P_{3m-1}, for m ≥ 1for the γ_r set D.

Case (ii) If $P_{it_l} = P_{3m}$, for m ≥ 1, 1 ≤ i ≤ n ,where l = 3,6,9,….. then the path P_{it_l} be P_l, where the vertex set of P_l, be $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$ for 1 ≤ i ≤ n, such that the vertex $u_{i,l1} = v_i$. Without loss of generality, choose the set of vertices $u_{i,l3}$, for 1 ≤ i ≤ n, l = 3,6,9,….. for theγ_r set D from the path of the type P_{3m} , for m ≥ 1, from each u_i then the set of vertices $u_{i,l2}$, for 1 ≤ i ≤ n, l = 3,6,9,…. and $u_{i,l4}$, for 1 ≤ i ≤ n, l≠3, where l = 6,9,12,….. are dominated. Since $\{u_{i,l1}\} = v_i$for 1 ≤ i ≤ n, l = 3,6,9,….. are already dominated by the set of vertices $\{u_{i,l2}\}$, for l =2, for 1 ≤ i ≤ n which is chosen for the γ_r set D from the path of the type P_{3m-1}, for m ≥ 1. Thus the two set of vertices $\{u_{i,l1}\} = v_i$for 1 ≤ i ≤ n, l = 3,6,9,….. and $\{u_{i,l2}\}$ for l = 3,6,9,….. , 1 ≤ i ≤ n are adjacent in V – D of G. Similarly choose the set of vertices $\{u_{i,l6}\}$ for 1 ≤ i ≤ n, l≠3, l = 6,9,12,….for the γ_r set D, then the set $\{u_{i,l5}\}$ for 1 ≤ i ≤ n, l≠3, l = 6,9,12,….. are dominated. Thus the sets $\{u_{i,l4}\}$and $\{u_{i,l5}\}$ for 1 ≤ i ≤ n, l≠3, l = 6,9,12,….. are adjacent in V – D. Proceeding similarly, choose the set of all vertices $\{u_{i,ll}\}$,l = 3,6,9,….., for the γ_r set D from the path of the type $P_{it_l} = P_{3m}$, for m ≥ 1, 1 ≤ i ≤ n ,l = 3,6,9,….. Let $C_2 = \{u_{i,l3}, u_{i,l6}, u_{i,l9}, \ldots, u_{i,ll}\}$ for 1 ≤ i ≤ n, l = 3,6,9,….. be the set of all vertices chosen for the γ_r set D.

Case (iii) If $P_{it_l} = P_{3m+1}$, for m ≥ 1, 1 ≤ i ≤ n , l = 4,7,10,….. then the path P_{it_l} be P_l. The vertex set of the path P_l, be $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$ for 1 ≤ i ≤ n, such that the vertex$u_{i,l1} = v_i$. Without loss of generality, choose the set of vertices as same as in the Case (ii). Thus choose the set of vertices $\{u_{i,l3}\}$, for 1 ≤ i ≤ n, l = 4,7,10,….. and $\{u_{i,l6}\}$, for 1 ≤ i ≤ n, l≠4, l = 7,10,13….. for the γ_r set D and similarly choose as same as in the case(ii) where the set of all vertices $\{u_{i,ll-1}\}$ and $\{u_{i,ll}\}$, for l = 4,7,10,….., are considered for the γ_r set D. Let $C_3 = \{u_{i,l3}, u_{i,l6}, u_{i,l9}, \ldots, u_{i,ll-1}, u_{i,ll}\}$ for 1 ≤ i ≤ n, l = 4,7,10,…., be the set of all vertices chosen for the γ_r set D from the path of the type $P_{it_l} = P_{3m+1}$, for m ≥ 1.

Thus the set $D = C_1 \cup C_2 \cup C_3$ where the set $D = \{u_{i,l2}, u_{i,l5}, u_{i,l8}, u_{i,l3}, u_{i,l6}, \ldots, u_{i,ll-1}, u_{i,ll}\}$, for 1 ≤ i ≤ n, l≥ 2 forms the γ_r set D of the unicyclic graph G, with cardinality $k = \gamma_r$. Thus G is k - γ_r enresdowed for any $k = \gamma_r$.

Consider any set D_1 of cardinality $k_1 = \gamma_r + 1$, then there exists the following cases

Case (iii)(a) Consider any set D_{11} of cardinality $k_{11} = \gamma_r + 1$, where the set $D_{11} = D \cup \{u_{i,lr}\}$, 1 ≤ i ≤ n, 2 ≤ l ≤ S_i, r ≥ 2, then there exists the following subcases.

Subcase (iii)(a_1) Consider a set $D_{11,1} = D \cup \{u_{i,lr}\}$, where 1 ≤ i ≤ n, 2 ≤ l ≤ S_i, and r = 2. Then the set of vertices $\{u_{i,l2}\}$ is adjacent to the $\{v_i\}$, 1 ≤ i ≤ n . By considering any vertices $\{u_{i,l2}\}$ which belongs to the V – D set , for obtaining the restrained dominating set, there exists no isolates in the set V – $D_{11,1}$. Thus the set $D_{11,1}$ forms the restrained dominating set of cardinality $k_{11,1} = \gamma_r + 1$, containing the minimum restrained dominating set D. Hence G is$k_{11,1}$ - γ_r enresdowed.

Subcase (iii)(a_2) Consider a set $D_{11,2} = D \cup \{u_{i,lr}\}$, where 1 ≤ i ≤ n, 2 ≤ l ≤ S_i, and r > 2, then the set of vertices $\{u_{i,lr}\}$ is not adjacent with any of the vertex $\{v_i\}$, 1 ≤ i ≤ n. Since the set of vertices $\{u_{i,lr}\}$, where 1 ≤ i ≤ n, 2 ≤

$l \leq S_i$, and $r > 2$, belongs to the path P_l, it is adjacent only to its adjacent vertices in P_l. Thus by considering this set of vertices $\{u_{i,lr}\}$ there exists an set of isolated vertices in the set $V - D_{11,2}$ and the set $D_{11,2}$ is not a restrained dominating set of cardinality $k_{11,2} = \gamma_r + 1$. Thus G is not $k_{11,2}$ - γ_r enresdowed.

Case (iii)(b) Consider any set $D_{12} = D \cup \{v_i\}, 1 \leq i \leq n$, which is of cardinality $k_{12} = \gamma_r + 1$. By considering any vertex $\{v_i\}, 1 \leq i \leq n$ from the cycle C_n, $n \geq 3$, then the vertex $\{u_{i,lr}\}$ for $l = 3$ and $r = 2$, $1 \leq i \leq n$ form an isolate vertex in $V - D_{12}$. Therefore the set D_{12} is not a restrained dominating set of G. Hence G is not k_{12} - γ_r enresdowed for any $k_{12} = \gamma_r + 1$. Thus G is k_1 - γ_r enresdowed for any $k_1 = \gamma_r + 1$.

Consider any set D_2 of cardinality $k_2 = \gamma_r + 2$, then there exists the following cases.
Case (iii)(b_1) Consider any set $D_{21} = D \cup \{v_{i_1}, v_{i_2}\}$, $1 \leq i_1, i_2 \leq n$, where the vertices v_{i_1}, v_{i_2} belong to the cycle C_n, $n \geq 3$. The cardinality of the set D_{21} is $k_{21} = \gamma_r + 2$. By considering any set of vertices v_{i_1}, v_{i_2} from the cycle C_n there exists an isolates in the set $V - D_{21}$. Hence the set D_{21} is not a restrained dominating set of G. Therefore G is not k_{21} - γ_r enresdowed.

Case (iii)(b_2) Consider the set $D_{22} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$.The set D_{22} is of cardinality $k_{22} = \gamma_r + 2$, then there exists the following subcases.

Subcase (iii)(b_{21}) Consider the set $D_{22,1} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, such that $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $l_1 = l_2$. Let $k_{22,1} = \gamma_r + 2$ be the cardinality of the set $D_{22,1}$.Thus the vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the same path, then there exists the following subcases.

Subcase (iii)($b_{21(1)}$) Consider the set $D_{22,11} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $l_1 = l_2$. If the given two vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} are adjacent in the same path. Then the set $V - D_{22,11}$ does not contain an isolate vertex. Thus the set $D_{22,11}$ forms an restrained dominating set containing the minimum restrained dominating set. Hence G is $k_{22,11}$ - γ_r enresdowed for any $k_{22,11} = \gamma_r + 2$.

Subcase (iii)($b_{21(2)}$) Consider the set $D_{22,12} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $l_1 = l_2$. If the vertices u_{i_1,l_1r_1}, u_{i_2,l_2r_2} are not adjacent in the same path, since these vertices belong to a path, a set of vertices adjacent to u_{i_1,l_1r_1}, u_{i_2,l_2r_2} in the path forms a set of isolates. Thus the set of isolate vertices exists in the set $V - D_{22,12}$. Thus the set $D_{22,12}$ is not a restrained dominating set of cardinality $k_{22,12} = \gamma_r + 2$. Hence G is not $k_{22,12}$ - γ_r enresdowed.

Subcase (iii)(b_{22}) Consider the set $D_{22,2} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, such that $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $l_1 \neq l_2$, then the following subcases exists.

Subcase (iii)($b_{22(1)}$) Consider the set $D_{22,21} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $i_1 = i_2$, then there exists the following subcases.

Subcase (iii)($b_{22(11)}$) Consider the set $D_{22,211} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$,where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $i_1 = i_2$, such that the vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the different paths and they are attached to the same v_i, $1 \leq i \leq n$ of C_n, $n \geq 3$, which result in non – existence of an isolate vertex in the set $V - D_{22,211}$. Hence the set $D_{22,211}$ forms a restrained dominating set containing the γ_r set D, with cardinality $k_{22,211} = \gamma_r + 2$. Therefore G is $k_{22,211}$ - γ_r enresdowed.

Subcase (iii)($b_{22(12)}$) Consider the set $D_{22,212} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$,where $1 \leq i_1, i_2 \leq n$, $2 \leq l_1, l_2 \leq S_{i1}$, $r_1, r_2 \geq 2$ and $i_1 = i_2$, then the vertices u_{i_1,l_1r_1}, u_{i_2,l_2r_2} which belong to different paths are not adjacent to any vertex v_i, $1 \leq i \leq n$ of C_n, $n \geq 3$,then there exists isolates in $V - D_{22,212}$. Thus the set $D_{22,212}$ is not a restrained dominating set of cardinality $k_{22,212} = \gamma_r + 2$. Therefore G is not $k_{22,212}$ - γ_r enresdowed for any cardinality $k_{22,212} = \gamma_r + 2$.

Subcase (iii)($b_{22(2)}$) Consider the set $D_{22,22} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 \ne i_2$. The vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the different paths, where the paths are attached to the differentv_i, $1 \le i \le n$.

Subcase (iii)($b_{22(21)}$) Consider the set $D_{22,221} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, $1 \le i_1, i_2 \le n, 2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 \ne i_2$, such that the verticesu_{i_1,l_1r_1} and u_{i_2,l_2r_2} are adjacent to the vertex v_i, $1 \le i \le n$. Then there exists no isolate in the set $V - D_{22,221}$. Hence the set $D_{22,221}$ forms a restrained dominating set containing the γ_r set D of G. The cardinality of the set $D_{22,221}$ be $k_{22,221} = \gamma_r + 2$. Hence G is $k_{22,221}$ - γ_r enresdowed.
Subcase (iii)($b_{22(22)}$) Consider the set $D_{22,222} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where the vertices u_{i_1,l_1r_1}, u_{i_2,l_2r_2} which belong to different paths of G and are not adjacent to v_i, $1 \le i \le n$, then there exists isolated vertices in the set $V - D_{22,222}$. Thus the set $D_{22,222}$ is not a restrained dominating set of G with cardinality $k_{22,222} = \gamma_r + 2$. Hence G is not $k_{22,222}$ - γ_r enresdowed.

Case (iv) Consider the set $D_{23} = D \cup \{v_{i_1}, u_{i,lr}\}$,$1 \le i_1, i \le n$, $2 \le l \le S_{i1}$, $r > 2$ and $\{v_i\}$, $1 \le i \le n$, be any vertex of the unicycle C_n, n≥ 3. By choosing any vertex from the cycle, the set of vertices adjacent to the vertex v_i form an isolated set of vertices, where the cardinality of the set D_{23} be $k_{23} = \gamma_r + 2$. Thus the set $V - D_{23}$ contains the isolated vertices and the set D_{23} is not a restrained dominating set. Thus G is not k_{23} - γ_r enresdowed and G is k_2 - γ_r enresdowed for any $k_2 = \gamma_r + 2$.

Case (v) Consider any set D_3 of cardinality $k_3 = \gamma_r + 3$, then the following subcases exists.
Case (v)(a) Consider the set D_{31},where $D_{31} = D \cup \{v_{i_1}, v_{i_2}, v_{i_3}\}$, $1 \le i_1, i_2, i_3 \le n$. By choosing the vertices v_{i_1}, v_{i_2},v_{i_3} from the unicycle C_n, n≥ 3 for the set D_{31}, the set $V - D_{31}$ contains a set of isolated vertices . Thus the set D_{31} is not a restrained dominating set of cardinality $k_{31} = \gamma_r + 3$. Hence G is not k_{31} - γ_r enresdowed.
Case (v)(b) Consider the set D_{32}, where the set $D_{32} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}, u_{i_3,l_3r_3}\}$, $1 \le i_1, i_2, i_3 \le n$, $2 \le l_1, l_2, l_3 \le S_{i1}$,$r_1, r_2, r_3 \ge 1$ of cardinality $k_{32} = \gamma_r + 3$, then there exists the following subcases.

Subcase (v)(b_1) Consider the set $D_{32,1}$, where $D_{32,1} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}, u_{i_3,l_3r_3}\}$,$1 \le i_1, i_2, i_3 \le n$,$2 \le l_1, l_2, l_3 \le S_{i1}$, $r_1, r_2, r_3 \ge 1$ and $r_1 = r_2 = r_3 \ne 1$. If these set of vertices u_{i_1,l_1r_1}, u_{i_2,l_2r_2} and u_{i_3,l_3r_3} are adjacent either to the same vertex v_i or different v_i, $1 \le i \le n$, then there exists no isolates in the set $V - D_{32,1}$. Thus the set $D_{32,1}$ forms a restrained dominating set containing the minimum restrained dominating set of cardinality $k_{32,1} = \gamma_r + 3$. Hence G is $k_{32,1}$ - γ_r enresdowed.

Subcase (v)(b_2) Consider the set $D_{32,2} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}, u_{i_3,l_3r_3}\}$, $1 \le i_1, i_2, i_3 \le n$, $2 \le l_1, l_2, l_3 \le S_{i1}$, $r_1, r_2, r_3 > 1$, where the set of vertices are chosen from the same path of the type P_{3m-1}, for m ≥ 1, where the paths P_{3m-1} are attached to same v_i or different v_i, $1 \le i \le n$ then there exists isolates in the set $V - D_{32,2}$. The set $D_{32,2}$ is not a restrained dominating set of cardinality $k_{32,2} = \gamma_r + 3$. Hence G is not $k_{32,2}$ - γ_r enresdowed.

Case (vi) Consider the set,D_{33} where $D_{33} = D \cup \{v_{i_1}, v_{i_2}, u_{i,lr}\}$, $1 \le i_1, i_2, i \le n$, $2 \le l \le S_i$, $r > 1$.The cardinality of the set D_{33} be $k_{33} = \gamma_r + 3$. Since the vertices of the cycle are chosen, the set D_{33} is not a restrained dominating set. Hence G is not k_{33} - γ_r enresdowed.

Case (vii) Consider the set $D_{34} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}, v_i\}$,$1 \le i_1, i_2, i \le n$, $2 \le l_1, l_2, \le S_{i1}$, $r_1, r_2 \ge 1$. Thus the existence of the vertex v_i of the cycle C_n, n≥ 3 in the set D_{34} results in the existence of isolates in the set $V - D_{34}$. Thus the set D_{34} is not a restrained dominating set of cardinality $k_{34} = \gamma_r + 3$. Hence G is not k_{34} - γ_r enresdowed.

Proceeding similarly, consider any set D_4 of cardinalit $k_4 = n + \left| \bigcup_{\substack{i=1 \\ 2 \le j \le S_i}}^{n} P_{it_j} - S_i \right| - 1$, then the set D_4 is not a restrained dominating set of G since there exists an isolate vertices in the set $V - D_4$. Hence G is not k_4 - γ_r enresdowed. Without loss of generality, consider a set D_5 of cardinality k_5 , where the cardinality k_5 is the union of the cardinality of the set of all vertices of cycle C_n, n≥ 3 and the cardinality of the set of all vertices in each path

$\{P_{it_j}\}$, $1 \leq i \leq n$ and $2 \leq j \leq S_i$, except the set of all vertices $\{v_i\}$,$1 \leq i \leq n$ of the cycle C_n . Therefore the cardinality k_5 is given by n+ $\left|\bigcup^{n}_{\substack{i=1 \\ 2 \leq j \leq S_i}} P_{it_j} - S_i\right|$. Hence G is k - γ_r enresdowed for any k , where $\gamma_r \leq k \leq$ n+ $\left|\bigcup^{n}_{\substack{i=1 \\ 2 \leq j \leq S_i}} P_{it_j} - S_i\right|$

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